

En route to fusion: confinement state as a wavelet

Antonina N. Fedorova, Michael G. Zeitlin

IPME RAS, St. Petersburg, V.O. Bolshoj pr., 61, 199178, Russia
<http://math.ipme.ru/zeitlin.html>, <http://mp.ipme.ru/zeitlin.html>

E-mail: zeitlin@math.ipme.ru, anton@math.ipme.ru

Abstract. A fast and efficient numerical-analytical approach is proposed for description of complex behaviour in non-equilibrium ensembles in the BBGKY framework. We construct the multiscale representation for hierarchy of partition functions by means of the variational approach and multiresolution decomposition. Numerical modeling shows the creation of various internal structures from fundamental localized (eigen)modes. These patterns determine the behaviour of plasma. The localized pattern (wavelet) is a model for energy confinement state (fusion) in plasma.

**“A magnetically confined plasma cannot
be in thermodynamical equilibrium”**

Unknown author ... Folklore

1. Introduction

It is well known that fusion problem in plasma physics could be solved neither experimentally nor theoretically during last fifty year, so it seems that there are the serious obstacles which prevent real progress in the problem of real fusion as the main subject in the area [1], [2]. Of course, it may be a result of some unknown no-go theorem(s) but it seems that the current theoretical level demonstrates that not all possibilities, at least on the level of theoretical and mathematical modeling, are exhausted. Definitely, the first thing which we need to change is a framework of generic mathematical methods which can help to improve the current state of the theory. Our postulates (conjectures) are as follows [3]:

A) The fusion problem (at least at the first step) must be considered as a problem inside the (non) equilibrium ensemble in the full phase space. It means, at least, that:

A1) our dynamical variables are partitions (partition functions, hierarchy of N-points partition functions),

A2) it is impossible to fix a priori the concrete distribution function and postulate it (e.g. Maxwell-like or other concrete (gaussian-like or even not) distributions) but, on the contrary, the proper distribution(s) must be the solutions of proper (stochastic) dynamical problem(s), e.g., it may be the well-known framework of BBGKY hierarchy of kinetic equations or something similar. So, the full set of dynamical variables must include partitions also.

B) Fusion state = (meta) stable state (with minimum entropy and zero measure) in the space of partitions on the whole phase space in which most of energy of the system is concentrated in the relatively small area (preferable with measure zero) of the whole domain of definition in the phase space during the time period which is enough to take reasonable part of it outside for possible usage. From the formal/mathematical point of view it means that:

B1) fusion state must be localized (first of all, in the phase space),
B2) we need a set of building blocks, localized basic states or eigenmodes which can provide
B3) the creation of localized pattern which can be considered as a possible model for plasma in a fusion state. Such pattern must be:

B4) (meta) stable and controllable, because of obvious reasons. So, the main courses are:

C1) to present smart localized building blocks which may be not only useful from point of view of analytical statements, such as the best possible localization, fast convergence, sparse operators representation, etc, but also exist as real physical fundamental modes,

C2) to construct various possible patterns with special attention to localized pattern which can be considered as a needful thing in analysis of fusion;

C3) after points C1 and C2 in ensemble (BBGKY) framework to consider some standard reductions to Vlasov-like and RMS-like equations (following the set-up from well-known results) which may be useful also. These particular cases may be important as from physical point of view as some illustration of general consideration.

The lines above are motivated by our attempts to analyze the hidden internal contents of the phrase mentioned in the epigraph of this paper: “A magnetically confined plasma cannot be in thermodynamical equilibrium.” Also, it should be noted that our results below can be applied to any scenario (fusion, ignition, etc): we describe pattern formation in arbitrary non-equilibrium ensembles.

2. Description

At this stage our main goal is an attempt of classification and construction of a possible zoo of nontrivial (meta) stable states/patterns: high-localized (nonlinear) eigenmodes, complex (chaotic-like or entangled) patterns, localized (stable) patterns (waveletons). We will use it later for fusion description, modeling and control. In our opinion localized (meta)stable pattern (waveleton) is the proper image for fusion state in plasma (energy confinement).

Our constructions can be applied to the hierarchy of N -particle distribution function, satisfying the standard BBGKY hierarchy (v is the volume):

$$\frac{\partial F_s}{\partial t} + L_s F_s = \frac{1}{v} \int d\mu_{s+1} \sum_{i=1}^s L_{i,s+1} F_{s+1}. \quad (1)$$

Our key point is the proper nonperturbative generalization of the previous perturbative multiscale approaches (like Bogolubov/virial expansions). The infinite hierarchy of distribution functions is:

$$\begin{aligned} F &= \{F_0, F_1(x_1; t), \dots, F_N(x_1, \dots, x_N; t), \dots\}, \\ F_p(x_1, \dots, x_p; t) &\in H^p, \quad H^0 = R, \quad H^p = L^2(R^{6p}), \\ F &\in H^\infty = H^0 \oplus H^1 \oplus \dots \oplus H^p \oplus \dots \end{aligned} \quad (2)$$

with the natural Fock space like norm (guaranteeing the positivity of the full measure):

$$(F, F) = F_0^2 + \sum_i \int F_i^2(x_1, \dots, x_i; t) \prod_{\ell=1}^i \mu_\ell. \quad (3)$$

Multiresolution decomposition (filtration) naturally and efficiently introduces the infinite sequence (tower) of the underlying hidden scales, which is a sequence of increasing closed subspaces $V_j \in L^2(R)$ [4]:

$$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \dots \quad (4)$$

Our variational approach [3] reduces the initial problem to the problem of solution of functional equations at the first stage and some algebraic problems at the second one. Let L be an arbitrary (non)linear differential/integral operator with matrix dimension d (finite or infinite), which acts on some set of functions from $L^2(\Omega^{\otimes n})$: $\Psi \equiv \Psi(t, x_1, x_2, \dots) = \left(\Psi^1(t, x_1, x_2, \dots), \dots, \Psi^d(t, x_1, x_2, \dots) \right)$, $x_i \in \Omega \subset \mathbf{R}^6$, n is the number of particles:

$$\begin{aligned} L\Psi &\equiv L(Q, t, x_i)\Psi(t, x_i) = 0, \\ Q &\equiv Q_{d_0, d_1, d_2, \dots}(t, x_1, x_2, \dots, \partial/\partial t, \partial/\partial x_1, \partial/\partial x_2, \dots, \int \mu_k) \\ &= \sum_{\substack{d_0, d_1, d_2, \dots \\ i_0, i_1, i_2, \dots = 1}} q_{i_0 i_1 i_2 \dots}(t, x_1, x_2, \dots) \left(\frac{\partial}{\partial t} \right)^{i_0} \left(\frac{\partial}{\partial x_1} \right)^{i_1} \left(\frac{\partial}{\partial x_2} \right)^{i_2} \dots \int \mu_k. \end{aligned} \quad (5)$$

Let us consider now the N mode approximation for the solution as the following ansatz:

$$\Psi^N(t, x_1, x_2, \dots) = \sum_{i_0, i_1, i_2, \dots = 1}^N a_{i_0 i_1 i_2 \dots} A_{i_0} \otimes B_{i_1} \otimes C_{i_2} \dots(t, x_1, x_2, \dots). \quad (6)$$

We shall determine the expansion coefficients from the following conditions:

$$\ell_{k_0, k_1, k_2, \dots}^N \equiv \int (L\Psi^N) A_{k_0}(t) B_{k_1}(x_1) C_{k_2}(x_2) dt dx_1 dx_2 \dots = 0. \quad (7)$$

As a result the solution has the following multiscale/multiresolution decomposition via nonlinear high-localized eigenmodes [4]:

$$\begin{aligned} F(t, x_1, x_2, \dots) &= \sum_{(i, j) \in Z^2} a_{ij} U^i \otimes V^j(t, x_1, x_2, \dots), \\ V^j(t) &= V_N^{j, \text{slow}}(t) + \sum_{l \geq N} V_l^j(\omega_l t), \quad \omega_l \sim 2^l, \\ U^i(x_s) &= U_M^{i, \text{slow}}(x_s) + \sum_{m \geq M} U_m^i(k_m^s x_s), \quad k_m^s \sim 2^m. \end{aligned} \quad (8)$$

So, we may move from the coarse scales of resolution (coarse graining) to the finest ones for obtaining more detailed information about the dynamical process. In this way one obtains contributions to the full solution from each scale of resolution or each time/space scale or from each nonlinear eigenmode. It should be noted that such representations give the best possible localization properties in the corresponding (phase) space/time coordinates. Numerical calculations are based on compactly supported wavelets and related wavelet families and on evaluation of the accuracy on the level N of the corresponding cut-off of the full system w.r.t. the norm (3):

$$\|F^{N+1} - F^N\| \leq \varepsilon \quad (9)$$

Numerical modeling shows the creation of various complex structures from localized modes, which are related to (meta)stable or unstable type of behaviour and the corresponding patterns (waveletons) formation (Figs. 1, 2). Reduced algebraic structure (7), Generalized Dispersion Relations, provide the pure algebraic control of stability/unstability scenario. So, we considered the construction for controllable (meta) stable wavelet configuration representing a reasonable approximation for the possible realizable confinement state.

3. Conclusions

Let us summarize our main results:

Physical Conjectures:

P1 State of fusion (confinement of energy) in plasma physics may and need be considered from the point of view of non-equilibrium statistical physics. According to this BBGKY framework looks naturally as first iteration. Main dynamical variables are partitions.

P2 Basic high localized nonlinear eigenmodes are real physical modes important for fusion modeling. Intermode multiscale interactions create various complex patterns from these fundamental building blocks, and determine the behaviour of plasma (Figs. 1, 2). High localized (meta) stable patterns (waveletons), considered as long-living fluctuations, are proper images for plasma in fusion state (Fig. 2).

Mathematical framework:

M1 The problems under consideration, like BBGKY hierarchies or their reductions are considered as pseudodifferential hierarchies in the framework of proper family of methods unified by effective multiresolution approach or local nonlinear harmonic analysis on the orbits of representations of hidden underlying symmetry of properly chosen functional space [3].

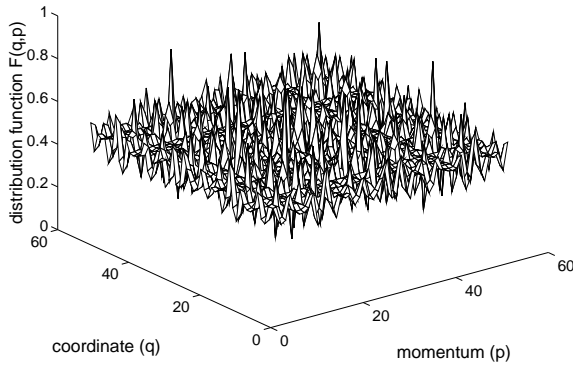


Figure 1. Trash: Chaotic Partition.

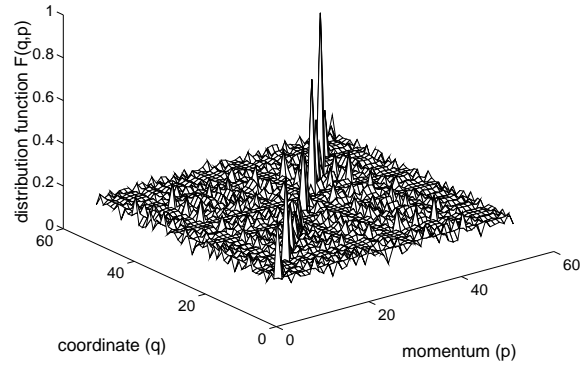


Figure 2. Goal: Waveleton/Fusion State.

M2 Formulas (8) based on Generalized Dispersion Relations (GDR) (7) provide exact multiscale representation for all dynamical variables (partitions, first of all) in the basis of high-localized nonlinear (eigen)modes. Numerical realizations in this framework are maximally effective from the point of view of complexity of all algorithms inside. GDR provide the way for the state control on the pure algebraical level.

Realizability:

According to this approach, it is possible on formal level, in principle, to control ensemble behaviour and to realize the localization of energy (confinement state) inside the waveleton configurations created from a few fundamental modes only during self-organization via possible (external) algebraical control (Figs. 1, 2) [3].

References

- [1] A. H. Boozer, Rev. Mod. Phys., 76, 1071 (2004).
- [2] R. C. Davidson and H. Qin, Physics of Intense Charged Particle Beams in High Energy Accelerators (World Scientific, Singapore, 2001); R. Balescu, Equilibrium and Nonequilibrium Statistical Mechanics, (Wiley, New York, 1975).
- [3] <http://math.ipme.ru/zeitlin.html> or <http://mp.ipme.ru/zeitlin.html>
- [4] Y. Meyer, *Wavelets and Operators* (Cambridge Univ. Press, 1990); F. Auger e.a., *Time-Frequency Toolbox* (CNRS, 1996); D. Donoho, *WaveLab* (Stanford, 2000)